Shape and Creep Deflection Analysis of Ice Dome
アイスドームの形状・クリープ変形解析

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概要

自重下において、経線方向と緯線方向の膜応力分布が指定された軸対称アイスドームの曲面形状と氷厚分布をシェルの膜理論に基づいて求める数値解析方法を示した。さらに、氷材料の歪速度と一軸応力の関係式が Glen の法則に従うと仮定して、そのアイスドームのクリープ変形量を不変量理論とシェルの膜理論に基づいて求める解析方法を展開した。以上の方法に基づいて、半開角 63.4°の部分球面を基準曲面形状とし、その中央点高さ／底面直径（＝0.309）を同一とする軸対称曲面を対象に数値計算を実施し、従来より用いられてきた球形曲面の場合に比べてクリープ変形を少なくして膜応力を小さくする非球形アイスドームの曲面形状と氷厚分布を例示した。この解析的検討と既往の 20～30m級球形アイスドームのフィールド実験結果及び 100 を超えるアイスシェルの建設経験は，目下進行中の東海大学・アイスパンテオンプロジェクトにおいて最終的に目指す 40m級の巨大アイスドームの実現に確信的根拠を与えるものと思われる。

Abstract
According to the shape and the creep deflection analysis of an axisymmetric ice dome, a non-spherical ice dome improves significantly the structural performance compared to the conventional type of spherical dome. The analysis of an axisymmetric ice dome is based on the followings: membrane theory for a thin shell and invariant theory for the ice obeyed Glen's law during the secondary creep stage. In addition to the numerical results of the analysis, the past construction experiences and the field experiments of 20-30m span ice domes would support the realization of a huge ice dome spanning 40 meters never existed before, which has almost same size as Pantheon in Rome well known as one of the biggest classical stone dome.

Keywords: non-spherical ice dome, shape and creep deflection analysis, Ice Pantheon

1 Introduction

Ice shells, which are thin curved plate-structures made of ice, are being used as winter structures in inland Hokkaido with sufficient snow and low temperature [1]. As the typical example of the applications, since 1997 in Tomamu, many ice shells are being used each winter for about 3 months as leisure-recreational facilities after skiing. The shell creates a beautiful space in the environment from the translucent thin plate and the unique curved surface shape. The interior space has a translucent atmosphere with full of natural light in daytime, and the exterior looks like a gigantic illuminator in the dark at night. The shell has also high structural efficiency, because the shape determined from the reticular geometry of the covered ropes on a pneumatic membrane follows automatically so that it works mainly compressive stress under self-weight load. Furthermore, the construction method of blowing snow and spraying water onto the pneumatic formwork has constructional rationality. Through these advantages, it is recognized that the ice shell is a practical ice structure for winter activities in inland Hokkaido.

According to the ice shell construction experiences so far, Ice shell has a tendency to creep easily with time even if the working stress is small. Large creep deformations end the dome's usability as an architectural structure, and cause instability leading to collapse. Therefore, it is very important to reduce the creep deformation for the durability of ice shells. The investigations have been carried out through the field experiments of ice domes with spans from 10 to 30-meters where the meteorological conditions such as outside air temperature, humidity, radiation, wind, snowfall and the accumulated load of snow on the dome vary [2][3][4][5]. The results led to a simplified formula that can predict the creep deformation of a spherical ice dome where the ice temperature is in the range of 0^{o}C to -5^{o}C [6].

This paper develops the shape and the creep deflection analysis of an axisymmetric ice dome based on the following assumptions: membrane theory for a thin shell and invariant theory for the ice obeyed Glen's law during the secondary creep stage [7][8][9]. The result shows that a non-spherical ice dome reduces significantly the creep deformations compared to that of the conventional type of spherical dome.

2 Theoretical consideration on shape and creep deflection

So far, a spherical cap is often used as the shape of ice dome. However, in the case of a large span such as Ice Pantheon, a large creep deformation may occur under gravity load. The large deformation is not good for the structural stability. "Form follows force" as one says, and a non-spherical shape might be able to reduce the amount of creep deformation. That might enhances the structural performance of the ice dome.

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2.1 Shape analysis

The shape analysis of a non-spherical ice dome is based on membrane theory for a thin shell [2]. Let us consider a dome of nonuniform thickness supporting its own weight. Fig.1 shows an element of a dome that is cut by two adjacent meridians and two parallel circles. The weight of the dome per unit area of the middle surface is ρh, and the two components of this weight along the coordinate axes are

$$p_\varphi = \rho h \sin\varphi \qquad p_n = \rho h \cos\varphi \tag{1}$$

Where ρ is the density of ice and h is the ice thickness.
The equation of equilibrium in the radial direction is

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = -p_n \tag{2}$$

Where N_φ, N_ϑ are the magnitudes of the membrane forces per unit length as shown in the Fig. 1. Let's assume that N_φ, N_θ are given in advance by Eq. (3).

$$N_\varphi = -\sigma_o h f_\varphi(\varphi), \qquad N_\theta = -\sigma_o h f_\theta(\varphi) \tag{3}$$

Where σ_0 is the compressive stress at the apex of the dome, and f_φ, f_θ are the distribution function of the membrane forces along the coordinate axes. f_φ, f_θ are functions of φ and 1 for φ=0. Substituting Eq. (1) and (3) into Eq.(2),

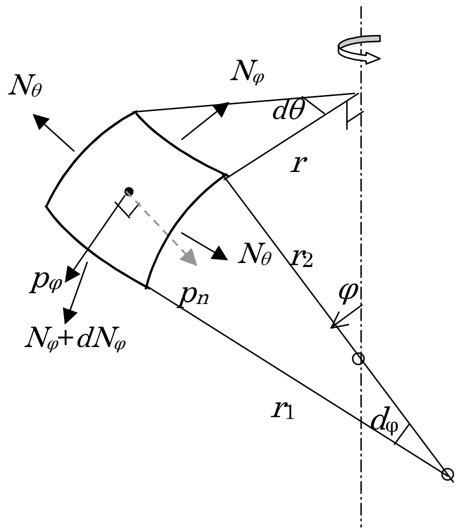

Fig.1 Membrane stress and external load

$$\sigma_o h\left(\frac{f_\varphi}{r_1} + \frac{f_\theta}{r_2}\right) = \rho h \cos\varphi \tag{4}$$

Using Eq. (4), $r = r_2 \sin\varphi \quad and \quad r_1 d\varphi = \dfrac{dr}{\cos\varphi}$ as shown in Fig.1, Eq.(5) forms a first order of differential equation for β.

$$\frac{d\beta}{d\varphi} = \frac{\beta f_\varphi \cos\varphi}{\beta\cos\varphi - f_\theta \sin\varphi}, \quad where \quad \beta = \alpha r, \quad \alpha = \frac{\rho}{\sigma_o} \tag{5}$$

Under the initial condition that $\dfrac{d\beta}{d\varphi} = 2 \quad for \quad \varphi \to 0,$ the numerical solution of Eq. (5) can be calculated by applying Runge-Kutta Method. Using the numerical value of β, z and r of the dome' shape are computed numerically based on Eq. (6)

$$\therefore \alpha(z - z_o) = \lim_{\varphi_o \to 0} \int_{\varphi_o}^{\varphi} \frac{\beta f_\varphi \sin\varphi}{\beta\cos\varphi - f_\theta \sin\varphi} d\varphi, \qquad \alpha r = \beta \tag{6}$$

Next, the thickness of the ice h is derived as follows. The equation of the equilibrium in the meridian direction is

$$\frac{d(N_\varphi r)}{d\varphi} - N_\theta r_1 \cos\varphi + p_\varphi r_1 r = 0 \tag{7}$$

Substituting $N_\varphi = -\sigma_o h f_\varphi, \quad N_\theta = -\sigma_o h f_\theta \quad and \quad p_\varphi = \rho h \sin\varphi$ into Eq. (7), Eq. (8) is obtained.

$$\frac{1}{Q}\frac{dQ}{d\varphi} = \frac{(f_\theta - f_\varphi)\cos\varphi + \beta\sin\varphi}{\beta\cos\varphi - f_\theta \sin\varphi}, \qquad where \quad Q = h f_\varphi \tag{8}$$

Solving Eq. (8) and placing Q=Q_o (=h_o) for φ=φ_o→0, h becomes Eq. (9).

$$\therefore \left(\frac{h}{h_o}\right) = \left(\frac{1}{f_\varphi}\right) \exp\left(\lim_{\varphi_0 \to 0} \int_{\varphi_0}^{\varphi} \left(\frac{\beta \sin\varphi + (f_\theta - f_\varphi)\cos\varphi}{\beta\cos\varphi - f_\theta \sin\varphi}\right) d\varphi\right) \tag{9}$$

Where h_o is the thickness at the apex of the dome.

2.2 Creep deflection

The creep model for the ice is assumed to be obeyed Glen's law during the secondary creep [3].

$$\frac{d\varepsilon}{dt} = \dot{\varepsilon} = k\sigma^n \tag{10}$$

Where σ is uniaxial stress, $\dot{\varepsilon}$ is uniaxial strain rate and k, n is constant. Strain-displacement relation is shown in Eq. (11).

$$\varepsilon_\theta = \frac{v\cos\varphi - w\sin\varphi}{r} = \frac{v\cot\varphi - w}{r_2}$$
$$\varepsilon_\varphi = \frac{v' - w}{r_1} \tag{11}$$

Where $' = \frac{d}{d\varphi}$, w is the displacement in the radial direction, v is the displacement in the meridian direction, as shown Fig.2, ε_θ is strain in the parallel direction and ε_φ is strain in the meridian direction. The relationship between the strain rate and the membrane stress is written in Eq. (12) by applying invariant theory [4].

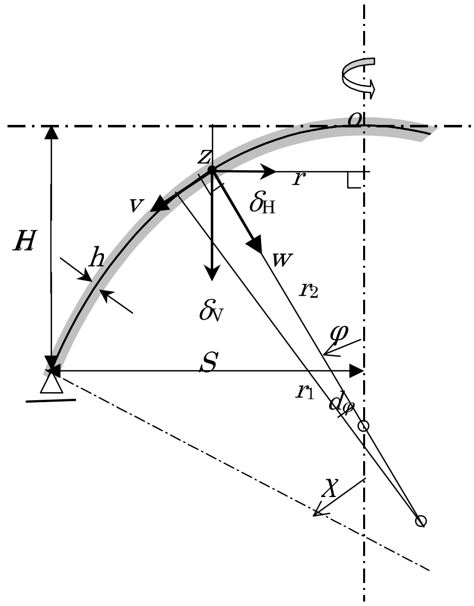

Fig.2 Displacement and geometry of dome

$$\dot{\varepsilon}_\varphi = k\left(\sigma_\varphi - \frac{1}{2}\sigma_\theta\right)\left(\sigma_\varphi^{\ 2} - \sigma_\varphi\sigma_\theta + \sigma_\theta^{\ 2}\right)^{\frac{n-1}{2}} = -k\sigma_o^{\ n}\left(f_\varphi - \frac{1}{2}f_\theta\right)\left(f_\varphi^{\ 2} - f_\varphi f_\theta + f_\theta^{\ 2}\right)^{\frac{n-1}{2}} \tag{12}$$
$$\dot{\varepsilon}_\theta = -k\sigma_o^{\ n}\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\ 2} - f_\varphi f_\theta + f_\theta^{\ 2}\right)^{\frac{n-1}{2}}, where \quad \sigma_\varphi = \frac{N_\varphi}{h} = \sigma_o f_\varphi, \quad \sigma_\theta = \frac{N_\theta}{h} = \sigma_o f_\theta$$

Eliminating w in Eq. (11) and using Eq. (12), Eq. (13) forms the first order of ordinary differential equation for $\dot{v}$.

$$\dot{v}' - \dot{v}\cot\varphi = r_1\dot{\varepsilon}_\varphi - r_2\dot{\varepsilon}_\theta = F(\varphi) \tag{13}$$
$$where \quad F(\varphi) = r_1\dot{\varepsilon}_\varphi - r_2\dot{\varepsilon}_\theta = k\sigma_o^{\ n} r\left(f_\varphi^{\ 2} - f_\varphi f_\theta + f_\theta^{\ 2}\right)^{\frac{n-1}{2}}\left\{-\frac{f_\varphi\left(f_\varphi - \frac{1}{2}f_\theta\right)}{\beta\cos\varphi - f_\theta\sin\varphi} + \frac{\left(f_\theta - \frac{1}{2}f_\varphi\right)}{\sin\varphi}\right\}$$

The general solution of Eq.(13) is $\dot{v} = \sin\varphi\left\{\int_0^\varphi \frac{F(\varphi)}{\sin\varphi} d\varphi + C\right\}$ (14)

Where C is a constant of integration to be determined from the condition at the support. Using the $\dot{v}$ in Eq. (14), $\dot{w}$ is written in Eq. (15)

$$\dot{w} = \dot{v}\cot\varphi - r_2\dot{\varepsilon}_\theta = \cos\varphi\left\{\int_0^\varphi \frac{F(\varphi)}{\sin\varphi} d\varphi + C\right\} + r_2 k\sigma_o^{\ n}\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\ 2} - f_\varphi f_\theta + f_\theta^{\ 2}\right)^{\frac{n-1}{2}} \tag{15}$$

The vertical displacement rate $\dot{\delta}_V$ shown in Fig. 2, is expressed by Eq. (16).

$$\dot{\delta}_V = \dot{v}\sin\varphi + \dot{w}\cos\varphi \tag{16}$$

The constant C is determined from the condition that for $\varphi=\chi$, the vertical displacement rate $\dot{\delta}_v$ is zero.

$$\therefore C = -\int_0^{\chi} \frac{F(\varphi)}{\sin\varphi} d\varphi - \bar{r}_2 k\sigma_o^{\,n}\left(\bar{f}_\theta - \frac{1}{2}\bar{f}_\varphi\right)\left(\bar{f}_\varphi^{\,2} - \bar{f}_\varphi \bar{f}_\theta + \bar{f}_\theta^{\,2}\right)^{\frac{n-1}{2}} \cos\chi \qquad (17)$$

Where $\overline{\nabla}$ is ∇ for $\varphi=\chi$.

Therefore, $\dot{\delta}_v$ and the horizontal displacement rate $\dot{\delta}_H = -\dot{v}\cos\varphi + \dot{w}\sin\varphi$ are written as follows.

$$\begin{aligned} \dot{\delta}_H &= -\dot{v}\cos\varphi + \dot{w}\sin\varphi = r_2 k\sigma_o^{\,n}\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}} \sin\varphi \\ \dot{\delta}_V &= \dot{v}\sin\varphi + \dot{w}\cos\varphi \\ &= -\int_\varphi^{\chi} \frac{F(\varphi)}{\sin\varphi} d\varphi + k\sigma_o^{\,n}\left\{ r_2\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}}\cos\varphi - \bar{r}_2\left(\bar{f}_\theta - \frac{1}{2}\bar{f}_\varphi\right)\left(\bar{f}_\varphi^{\,2} - \bar{f}_\varphi \bar{f}_\theta + \bar{f}_\theta^{\,2}\right)^{\frac{n-1}{2}}\cos\chi \right\} \end{aligned} \qquad (18)$$

And then substituting

$$\beta = \alpha r, \quad \alpha = \frac{\rho}{\sigma_o}, \qquad F(\varphi) = r_1\dot{\varepsilon}_\varphi - r_2\dot{\varepsilon}_\theta = rk\sigma_o^{\,n}\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}}\left\{ -\frac{f_\varphi\left(f_\varphi - \frac{1}{2}f_\theta\right)}{\beta\cos\varphi - f_\theta\sin\varphi} + \frac{\left(f_\theta - \frac{1}{2}f_\varphi\right)}{\sin\varphi} \right\}$$

$$and \quad E(\varphi) = \frac{\alpha}{k\sigma_o^{\,n}}\frac{F(\varphi)}{\sin\varphi} = \beta\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}}\left\{ -\frac{f_\varphi\left(f_\varphi - \frac{1}{2}f_\theta\right)}{\sin\varphi(\beta\cos\varphi - f_\theta\sin\varphi)} + \frac{\left(f_\theta - \frac{1}{2}f_\varphi\right)}{\sin^2\varphi} \right\}$$

into Eq.(18), $\dot{\delta}_H$ and $\dot{\delta}_v$ are written in Eq. (19).

$$\begin{aligned} \therefore \alpha\dot{\delta}_H &= k\sigma_o^{\,n}\beta\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}} \\ \therefore \alpha\dot{\delta}_V &= k\sigma_o^{\,n}\left[\left\{\frac{\beta}{\sin\varphi}\left(f_\theta - \frac{1}{2}f_\varphi\right)\left(f_\varphi^{\,2} - f_\varphi f_\theta + f_\theta^{\,2}\right)^{\frac{n-1}{2}}\cos\varphi - \frac{\bar{\beta}}{\sin\chi}\left(\bar{f}_\theta - \frac{1}{2}\bar{f}_\varphi\right)\left(\bar{f}_\varphi^{\,2} - \bar{f}_\varphi\bar{f}_\theta + \bar{f}_\theta^{\,2}\right)^{\frac{n-1}{2}}\cos\chi\right\} - \int_\varphi^{\chi} E(\varphi)d\varphi\right] \end{aligned} \qquad (19)$$

The displacement rate at the apex $\dot{\delta}_{Vtop}$ is

$$\therefore \alpha\dot{\delta}_{V\,top} = k\sigma_o^{\,n}\left\{ \begin{array}{l} 1 - \bar{\beta}\left(\bar{f}_\theta - \frac{1}{2}\bar{f}_\varphi\right)\left(\bar{f}_\varphi^{\,2} - \bar{f}_\varphi\bar{f}_\theta + \bar{f}_\theta^{\,2}\right)^{\frac{n-1}{2}}\dfrac{\cos\chi}{\sin\chi} \\ -\int_0^{\chi} E(\varphi)d\varphi \end{array} \right\} \qquad (20)$$

The average vertical displacement rate $\dot{\delta}_{Vav}$ is

$$\dot{\delta}_{V\,av} = \frac{\int_0^{\chi}\dot{\delta}_V(2\pi r r_1 d\varphi)}{\int_0^{\chi} 2\pi r r_1 d\varphi} = \frac{\int_0^{\chi}\dot{\delta}_V r r_1 d\varphi}{\int_0^{\chi} r r_1 d\varphi} \qquad (21)$$

2.3 Numerical results for Ice Pantheon Dome

The stress distribution of the membrane forces is given in the following.

$$f_\varphi = \frac{2}{1+\cos(x\chi)}, f_\theta = 2\left(\cos(x\chi) - \frac{1}{1+\cos(x\chi)}\right),$$

$$Where \quad x = \frac{\varphi}{\chi}, \chi = 63.435^o$$

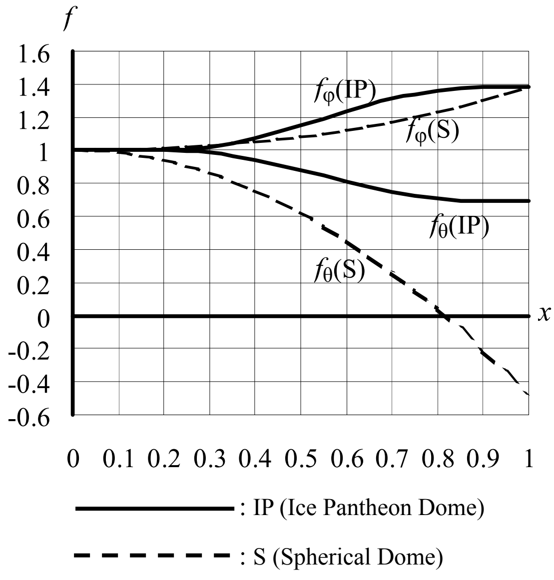

Fig. 3 Distribution of membrane stress

The numerical results show a spherical dome of uniform thickness, called S dome. In the case of Ice pantheon dome (IP dome), the following stress distributions are adopted under the condition of the same rise/(base diameter) as the S dome above mentioned.

$$\varepsilon_\theta = 0 \quad for \quad \varphi = \chi, \quad \therefore f_\theta - \frac{1}{2} f_\varphi = 0 \quad for \quad \varphi = \chi (= 55.502^\circ), \quad x = \frac{\varphi}{\chi}$$

$$f_\varphi = \begin{cases} 1, & 0 < x < x_o \\ 1 + \frac{k_f}{2}\left(1 + \sin\left(\frac{(x - 0.5(x_1 + x_o))\pi}{(x_1 - x_o)}\right)\right), & x_o < x < x_1 \\ 1 + k_f, & x_1 < x < 1 \end{cases}, \qquad x_o = 0.2, x_1 = 0.9,$$

$$k_f = 2\alpha - 1 = 2 \times 0.69098 - 1 = 0.38196, \quad \alpha = \frac{1}{1 + \cos 63.435^o} = 0.69098$$

$$k_t = \frac{(1 - k_f)}{2} = \frac{(1 - 0.38196)}{2} = 0.30902, \qquad f_t = \begin{cases} 1, & 0 < x < x_o \\ 1 - \frac{k_t}{2}\left(1 + \sin\left(\frac{(x - 0.5(x_1 + x_o))\pi}{(x_1 - x_o)}\right)\right), & x_o < x < x_1 \\ 1 - k_f, & x_1 < x < 1 \end{cases}$$

Fig. 4 and Fig. 5 show the numerical results of these shape analysis. In addition, the creep displacement of IP dome is very small compared to that of S dome as shown in Table 1. From these results, IP dome improves significantly the structural performance such as creep deflection and magnitude of stress compared to S dome. In the case of IP dome spanning 40 m, σ_{OIP} is 7.5 N/cm^2 under its own weight using β = 2.123, ρ=0.85 g/cm^3 and r=20 m in Eq. (5). The value of the σ_{OIP} corresponds to about 1/50th of the uniaxial compressive strength of ice. Therefore, the construction of IP dome spanning 40m has enough strength to stand theoretically.

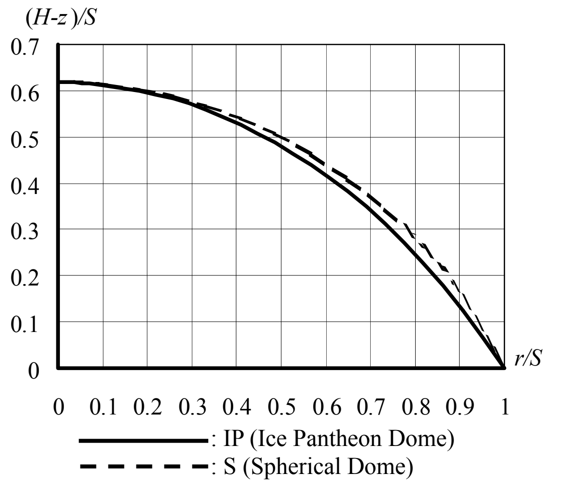

Fig. 4 Meridian curve of dome

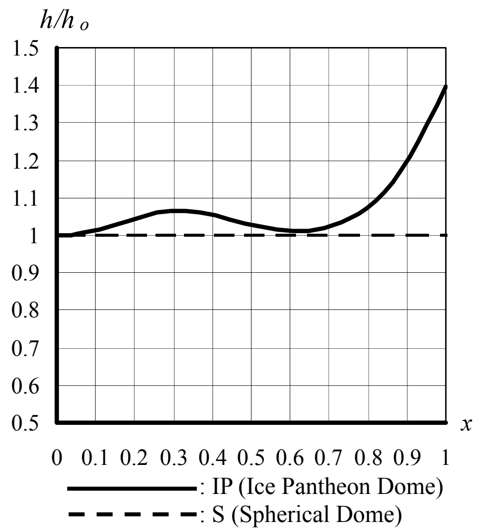

Fig. 5 Distribution of thickness

Table 1 Comparisons of displacement rate and stress between S and IP dome

n (see Eq.(10))	$(\delta_{VIP}/\delta_{VS})_{average}$	$(\delta_{VIP}/\delta_{VS})_{apex}$	$(\sigma_{OIP}/\sigma_{OS})$
1	0.388	0.463	
2	0.263	0.344	0.842
3	0.172	0.244	

δ_{VIP}: vertical displacement rate of IP dome, δ_{VS}: vertical displacement rate of S dome
$(\delta_{VIP}/\delta_{VS})_{average}$: Ratio of average δ_{VIP} to average δ_{VS}, $(\delta_{VIP}/\delta_{VS})_{apex}$: Ratio of δ_{VIP} at apex to δ_{VS} at apex
σ_{OIP}: σ_O in IP dome, σ_{OS}: σ_O in S dome

3 Ending remarks

According to the shape and the creep deflection analysis of an axisymmetric ice dome based on membrane theory for a thin shell and invariant theory for the ice obeyed Glen's law during the creep, the non-spherical ice dome improves significantly the structural performance compared to the conventional type of spherical dome. In addition to the numerical result of the analysis in this paper, the past construction experiences and field experiments of 20-30m span ice domes would support the realization of a huge ice dome spanning 40 meters never existed before, which has almost same size as Pantheon in Rome well known as one of the biggest classical stone dome. The ice dome is easier to construct than stone dome and the strength/density of the ice is almost same as that of stone in short term loading, so it could be possible for students as amateur to construct a 40m-span ice dome if they gradually experience the construction from small domes. Towards the realization of the ice dome, so called 'Ice Pantheon', the students of Tokai University started to go on an exciting, thrilling and wonderful voyage under the technical guidance by the authors. In winter of 2009, as the first step toward this end, a small size of 10-m span ice dome was constructed. And then, in last winter of 2010, the students constructed a non-spherical 15-m span ice dome [10].

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